Midterm 1 Content

Insertion Sort w/ Sentinel

Code

```
A[0] ← -∞
for i ← 2 to n do
    t ← A[i]
    j ← i - 1
    while A[j] > t do
        A[j+1] ← A[j]
        j ← j-1
    A[j+1] ← t
```

Comparisons

$$B = \sum_{i=2}^{n} 1$$

$$W = \sum_{i=2}^{n} i$$

$$A = \sum_{i=2}^{n} \frac{1}{i} \sum_{j=1}^{i} j$$

Exchanges

$$B = 0$$

$$W = \sum_{i=2}^{n} i$$

$$A = \sum_{i=2}^{n} \frac{1}{i} \sum_{j=1}^{i} j$$

Insertion Sort w/out Sentinel

Code

```
for i ← 2 to n do
    t ← A[i]
    j ← i - 1
    while j > 0 && A[j] > t do
        A[j+1] ← A[j]
        j ← j-1
    A[j+1] ← t
```

Comparisons

$$B = \sum_{i=2}^{n} 1$$

$$W = \sum_{i=2}^{n} (i-1)$$

$$A = \sum_{i=2}^{n} \frac{1}{i-1} \sum_{j=1}^{i} j$$

Exchanges

Same as with Sentinel

Bubble Sort

Code

```
for i ← n downto 2 do
    for j ← 1 to i - 1 do
        if A[j] > A[j+1] do
            A[j] ⟷ A[j+1]
```

Comparisons

$$\sum_{i=2}^{n} \sum_{j=1}^{i-1} 1 = \frac{(n-1)(n)}{2}$$

Exchanges

$$B = 0$$

$$W = \frac{(n-1)(n)}{2}$$

$$A = \sum_{i=2}^{n} \sum_{j=1}^{i-1} 0.5$$

good luck

Max Continuous Sum

Code

```
M ← 0; S ← 0
for i ← 1 to n do
    S ← max(S+A[i], 0)
    M ← max(M, S)
```

Comparisons

$$n$$

Exchanges

$$0$$

Sift for Max-Heap

Code

```
func sift(A, i, n) do
    l ← 2i     // Child1
    r ← 2i + 1 // Child2
    M ← i      // Maximum
    if l ⩽ n && A[l] > A[i] do
        M ← l
    if r ⩽ n && A[r] > A[i] do
        M ← r
    if M ≠ i do
        A[i] ⟷ A[M]
        sift(A, M, n)
    return A
```

Comparisons (i=0)

$$B = 2$$

$$W = 2\lg(n)$$

Exchanges (i=0)

$$B = 0$$

$$W = \lg(n)$$

Max-Heapify Random Array

Code

```
for i ← ⌊n/2⌋ downto 1 do
    sift(A, i, n)
```

Comparisons

$$B = \sum_{i=1}^{\lfloor n/2 \rfloor} 2 = n$$

$$W = \sum_{i=1}^{\lfloor n/2 \rfloor} 2\lg(i) \leq 2\,n\lg(n)$$

Exchanges

$$B = 0$$

$$W = \sum_{i=1}^{\lfloor n/2 \rfloor} \lg(i) \leq n\lg(n)$$

Sort Max-Heap

Code

```
for i ← n downto 2 do
    A[i] ⟷ A[1]
    sift(A, 1, n)
```

Comparisons

$$B = \sum_{i=2}^{n} 2 = 2(n-1)$$

$$W = \sum_{i=2}^{n} 2\lg(n) = 2(n-1)\lg(n)$$

Exchanges

$$B = 0$$

Funny Log Thing:

$$(a^x)^{log_a(n)} = n^x$$

Combination & Permutations

Combinations with Repetition:

$${}^{n}C_{r} = \frac{(n+r-1)!}{r!(n-1)!}$$

Permutations with Repetition:

$${}^{n}P_{r} = n^{r}$$

Combinations without Repetition:

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Permutations without Repetition:

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Midterm 2 Content

Quicksort

Code

```
func quicksort(s, e, A[]) do
    if s > e do
        exit func
    pi ← (s ... e).rand() or approx_median(A)
    A[pi] ⟷ A[s] // Move pivot to start
    q ← partition(s, e, A)
    quicksort(s, q-1, A)
    quicksort(q, e, A)
```

Recurrences (Random Pivot)

$W: T(n) = T(n-1) + T(0) + n - 1$

$B: T(n) = 2T(\lfloor n/2 \rfloor) + n - 1$

$A: T(n) = \frac{1}{n}\sum_{q=1}^{n}(T(n-q) + T(q-1)) + n - 1$

Selection

Code

```
func selection(s, e, k, A[]) do
    if s ⩾ e do
        exit func
    pi ← (s ... e).rand() or approx_median(A)
    A[pi] ⟷ A[s] // Move pivot to start
    q ← partition(s, e, A)
    if k < q do
        selection(s, q-1, k, A)
    else if k > q do
        selection(q, e, k, A)
    else
        return A[k]
```

Recurrences (Random Pivot)

$W: T(n) = T(n-1) + n - 1$

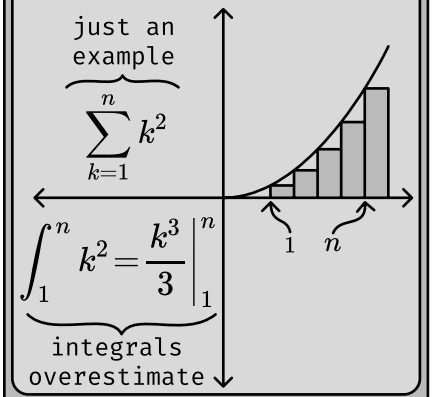
$B: T(n) = T(\lfloor n/2 \rfloor) + n - 1$

$A: T(n) = \frac{1}{n}\sum_{q=1}^{n}(T(n-q)) + n - 1$

you got this queen

Bounds and Sums

Graph



Even Splitting

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n/2} k^2 + \sum_{k=n/2+1}^{n} k^2$$

Optimal Splitting

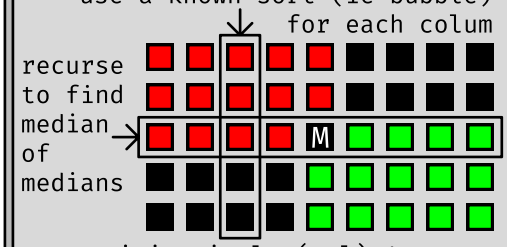
$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{s} k^2 + \sum_{k=s+1}^{n} k^2$$

Expand and derivate with respect to s; set result equal to 0 to find local minima / maxima / optimal bounds.

Methodology

In all of these cases, replace the value being summed with the lower or upper bound to obtain approximations.

Approximate Median



Key:	Size:
⩽ median	$\lfloor \frac{n}{10} \rfloor * 3 \approx \frac{3n}{10}$
⩾ median	$\lfloor \frac{n}{10} \rfloor * 3 \approx \frac{3n}{10}$
unkown	$\lfloor \frac{n}{10} \rfloor * 2 * 2 \approx \frac{4n}{10} = \frac{2n}{5}$

Karatsuba's Fast Multiplication Algorithm

Mathematics

ab and cd are numbers of n digits

$ab * cd =$

$(10^{n/2}a + b)(10^{n/2}c + d) =$

$10^n ac + 10^{n/2}(ad + bc) + bd$

$10^n\ ac + bd$

TTTT0000

+0000UUUU

TTTTUUUU

(no addition)

$10^{n/2}(\ ad + bc\ )$

00VVVV00

+00WWWW00

00XXXX00

αn additions

TTTTUUUU

+00XXXX00

TTYYYYUU

αn additions

Recurrence

$T(n) = 4T(n/2) + 2\alpha n$

$T(1) = \mu$

Solution

$$\underbrace{\sum_{i=0}^{lg(n)-1}(4^i(2\alpha\frac{n}{2^i}))}_{\text{tree body}} + \underbrace{\mu 4^{\lg(n)}}_{\text{leaves}}$$

Midterm 2 Content - Continued

Merge

Code

```
func merge(A[], B[]) do
    C ← []
    while len(A) > 0 && len(B) > 0 do
        if A.head() ⩽ B.head() do
            C.append(A.head())
            A.drophead()
        else do
            C.append(B.head())
            B.drophead()

    while len(A) > 0 do
        C.append(A.head())
        A.drophead()
    while len(B) > 0 do
        C.append(B.head())
        B.drophead()

    return C
```

Comparisons (len(A)=len(B)=n/2)

$$B = \sum_{i=1}^{n/2} 1 = n/2$$

$$W = \sum_{i=1}^{n-1} 1 = n - 1$$

Mergesort

Code

```
func mergesort(s, e, A[]) do
    if s ⩾ e do
        exit func
    c = ⌊(s+e)/2⌋
    X ← mergesort(s, c, A[])
    Y ← mergesort(c+1, e, A[])
    return merge(X, Y)
```

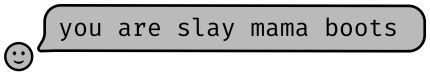
Recurrences

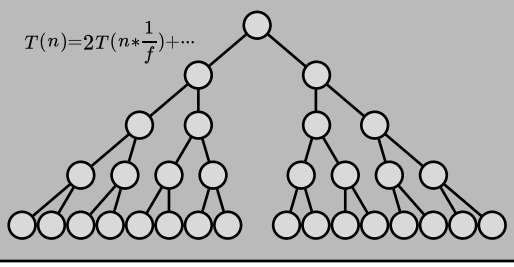
$B : T(n) = 2T(n/2) + n/2$

$W : T(n) = 2T(n/2) + n - 1$

Solution (worst case)

$$\sum_{i=0}^{\lg(n)-1} 2^i(\frac{n}{2^i} - 1) = n\lg(n) - n + 1$$



Layer	Tree Method	
0		Body
1		
2		
$\log_f(n) - 1$		
$\log_f(n)$		Leaves

Bucketsort

Code

```
func bucketsort(A[]) do
    for i in (1 ... n) do
        solve A[i]'s bucket
        store A[i] in bucket
    concatenate bucket contents
```

Comparisons & Exchanges

0

Radixsort

Code

```
func radixsort(A[]) do
    for each digit
        bucketsort(digit)
```

Comparisons & Exchanges

0

Countingsort

Code (values in (0 ... k-1))

```
func countingsort(A[]) do
    C = array of k zeros
    for i in (1 ... n) do
        C[A[i]] += 1
    t ← 0
    for i in (0 ... k-1) do
        C[i] ← C[A[i]] + 1

    for j in (n ... 1) do
        B[C[A[j]]] ← A[j]
        C[A[j]] ← C[A[j]] - 1

    return B
```

Comparisons & Exchanges

0

Final Content

Dijkstra

Goal

Discover the shortest path between an origin point and all other nodes in a weighted graph. AKA The Shortest Path Tree.

Code

```
func dijkstra(G, s) do
    dist ← [∞, n times]
    pred ← [NULL, n times]
    S ← []
    dist[s] ← 0
    while len(S) ≠ n do
        x ← vertex in G not including S with lowest weight
        for each edge attached to x do
            y ← the connected vertex
            if dist[x] + edge.weight < dist[y] do
                dist[y] ← dist[x] + (Weight of Edge x,y)
                pred[y] ← x
        S.append(x)

    return pred
```

Adjacency Matrix

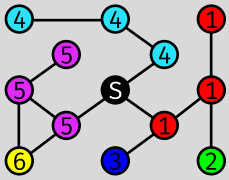
$$T(G) = \Theta(V(2V)) = \Theta(V^2)$$

Adjacency List

$$T(G) = \Theta(V(V+E)) = \Theta(VE)$$

Depth First Search

Graph



Properties

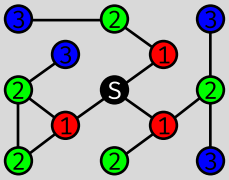
```
visited = [False, n times]
function DFS(G,x):
    visited[x] ← True
    for each vertex y adjacent x do
        if visited[y] == False do
            DFS(G,y)
DFS(G,s)
```

Run Time

$$O(V) + \Theta(V) + \Theta(E) = O(V+E)$$

Breadth First Search

Graph



Code

```
func BFS(G, s) do
    // Unprocessed Vertices
    queue ← [s]
    // Discovered Vertices
    D ← [False, n times]
    D[s] ← TRUE
    while len(queue) > 0 do
        u = queue.pop()
        for each vertex v adjacent u do
            if D[v] == False do
                queue.push(v)
                D[v] ← True
                // Check something
```

Run Time

$$O(V) + \Theta(V) + \Theta(E) = O(V+E)$$

Prim

Goal

Discover the Minimum Spanning Tree (MST) of a weighted graph.

Code

```
func prim(G) do
    let s ← arbitrary starting vertex
    T ← []
    U ← [s]
    while U ≠ V do
        (u, v) ← lowest cost edge where u ∈ U and v ∈ (V - U)
        T ← T ∪ (u, v)
        U ← U ∪ v
```

Adjacency Matrix

$$T(G) = O(V^2)$$

Min-Heap for Edges

$$T(G) = O(V \lg(E))$$

Kruskal

Goal

Discover the Minimum Spanning Tree (MST) of a weighted graph.

Code

```
func kruskal(G) do
    let EX ← the edges of the graph, sorted
    T ← []
    for each edge (u, v) in EX do
        if u and v are not part of the same tree do
            T.append((u, v))
```

Runtime

For a graph with E edges and V vertices, Kruskal's algorithm can be shown to run in O(E log E) time, or equivalently, O(E log V) time, all with simple data structures.

Final Content

Parallel Analysis

Speedup and Efficiency

$$S_P(N) = \frac{T_1(N)}{T_P(N)}$$

$$E_P(N) = \frac{S_P(N)}{P} = \frac{T_1(N)}{PT_P(N)}$$

An algorithm is only considered "Efficient" iff

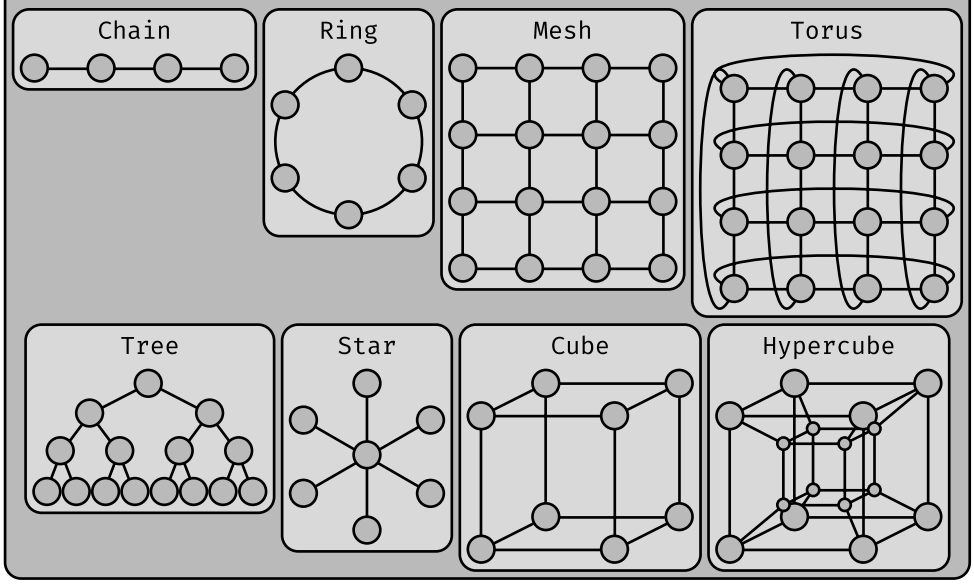
$$E_P(N) = \Theta(1)$$

Note:

$$1 \leq S_p(N) \leq P$$

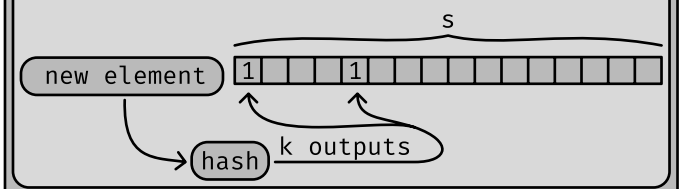
$$\frac{1}{P} \leq E_P(N) \leq 1$$

Parallel Computing Models



Bloom Filters

Visualization



Analysis (x=existing elements)

Bit is Zero

$$(1 - 1/s)^{kx}$$

Zobrist Hashing

Methodology

Assign a unique bit string identifier to each unique board piece. When the state of one board piece changes, XOR it's existing position with the existing hash to remove its existing position from the hash by the magical property of XOR. Then, XOR its new position with the hash to get the new board state. Setup may be expensive but all future board moves will run in constant time (2 * XOR).

Random Numbers

Linear Congruential Generator

This methodology is limited because a, c, and m must all be chosen very carefully. Even still, the tree of possible outcomes may not be evenly weighted, leading some outputs to be more prevalent than others by necessity.

$$x_{n+1} = ax_n + c \bmod m$$

Middle Square

Simply square a number and use its middle digits as the seed for the next number in the sequence.

$$x_{n+1} = \left(\frac{x_n^2}{2^{\frac{d}{2}}}\right) \mod d$$

NP

Show NP

To show that an algorithm is in NP, the simplest methodology is to show that verifying a possible solution, called a 'certificate', can be done in polynomial time.

Decision Version

The decision version of an optimization problem should be a task of comparable computational complexity, yet only **outputs** a Yes/No response. It's input is identical to the optimization version, seldom a variable k which represents the axis being optimized.
To show that decision follows from optimization time, use the output of the optimization problem, determine the k associated with it, knowing this is the optimal k. Finally showcase that this output is correct for the given k.

Optimization Version

The optimization verson of a problem **outputs** the actual solution in question. Its input is identical to decision, seldom the variable k which is absent.
To show that optimization follows from decision, use the output of the decision problem over and over to determine the optimal k. Then, run the decision version with that constant k and a modified input over and over until the true solution can be deduced.